Homogeneous Dynamics: An Introduction to Ratner's Theorems on Unipotent Flows

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Abstract

A simple introduction of what is called today as Ratner's theorems on unipotent flows with a classical application to Number Theory.

1 Introduction: Homogeneous Spaces

We start with some basic definitions and concepts concerning a sort of pragmatic treatment to this very introduction of what can be called *Homogeneous Dynamics*.

1.1 Actions

Let's begin with some terminations about actions.

Definition 1. (**Left actions**). A *left action* of a Lie group G on a manifold M is a map

$$\begin{aligned} a :& G \times M \to M \\ & (g,p) \mapsto g.p := a_g(p) \end{aligned}$$

such that $g_1(g_2.p) = (g_1.g_2).p$ and $e.p = p$, where $g_1, g_2 \in G$, $m \in M$ with e being the identity in G.

Definition 2. (**Basic notions on group actions**)

- An action is said to be **smooth** if the defining map of the action is smooth.
- The **orbit** of $p \in M$ under the action of G is defined as:

$$G.p = \{g.p : g \in G\}.$$

- The **isotropy group** of $p \in M$ is defined as

$$G_p = \{g \in G : g.p = p\}.$$

- The action is said to be **free** if

$$G_p = \{e\} \text{ for all } p \in M.$$

- A continuous action is said to be **proper** if the map

$$\begin{aligned} G \times M &\to M \times M \\ (g, p) &\longmapsto (g.p, p) \end{aligned}$$

  is a proper map. Where proper map is defined as a map between topological spaces such that the preimage of any compact subset is compact itself.

- An action is said to be **transitive** if for every $p, q \in M$ there exist $g \in G$ such that $q = g.p$.

1.2 Homogeneous Spaces

Definition 3. A **homogeneous space** is a smooth manifold endowed with a *transitive*, *smooth* action by a Lie group.

Example (Euclidean Group): Consider the group $\xi(n) := \mathbb{R}^n \times O(n)$ with the group multiplication $(b, A).(b', A') = (b + Ab, A.A')$ for every $b, b' \in \mathbb{R}^n$ and $A, A' \in O(n)$, where $O(n)$ is the n-dimensional orthogonal group.
$\xi(n)$ acts on $\mathbb{R}^n$ via the action

$$\begin{aligned} \xi(n) \times \mathbb{R}^n &\to \mathbb{R}^n \\ ((b, A), x) &\longmapsto b + Ax. \end{aligned}$$

Since we can obtain any vector from any other just by linear transformations in $\mathbb{R}^n$, the action is transitive. Then, $\mathbb{R}^n$ becomes a homogeneous space under the action of $\xi(n)$. □

Now, we are presenting an important result concerning characterization and construction of homogeneous spaces.

Theorem 4. ***(Quotient manifold theorem)*** *Suppose G is a Lie group acting smoothly, transitively and properly on a smooth manifold M. Then the orbit space G/M is a topological manifold with a unique smooth structure such that the quotient map $\pi' : M \to G/M$ is a smooth submersion.*

Proof. See theorem 5.10 in [4]. □

Theorem 5. ***(Homogeneous space construction)*** *Let G be a Lie group and H a closed subgroup of G.*

1. *The left coset space G/H is a topological manifold of dimension $\dim(G) - \dim(H)$ and has a unique smooth structure such that the quotient map $\pi : G \to G/H$ is a smooth submersion.*

2. *The left action of G on G/H given by*

$$g_1.(g_2H) = (g_1.g_2)H$$

turns G/H into a homogeneous space.

Proof. See example 5.11 in [4]. □

2 The Orbit Closure Theorem

We begin with some introduction to understand the idea behind the proper statement of the main theorem.

2.1 What is The Orbit Closure Theorem?

Let's start with some motivational example to introduce the ideas behind unipotent flows.

Consider the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim$, where the $\sim$ denotes the equivalence relation defined by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}^n$. Each element of $\mathbb{T}^n$ is an equivalence class

$$[x] = \{x + y : y \in \mathbb{Z}^n\} \text{ or } [x] = x + \mathbb{Z}^n.$$

Any vector $v \in \mathbb{R}^n$ define a flow ϕ_t on $\mathbb{T}^n$ by

$$\phi_t([x]) = [x + tv] \text{ for every } t \in \mathbb{R}.$$

It is a well known fact that the closure of the orbit of every point $[x]$ is a subtorus of $\mathbb{T}^n$. (See [5], for example). Being more precise, this is the same as saying that there exist a vector subspace S of $\mathbb{R}^n$ such that:

1. $v \in S$ (and from this, the entire orbit ϕ_t of $[x]$ is contained in $[x + S]$).
2. the image $[x + S]$ is compact and
3. the ϕ_t-orbit of $[x]$ is dense in $[x + S]$, that is, $[x + S]$ is the closure of the orbit of $[x]$.

In short, the closure of every orbit is a nice geometric subset of $\mathbb{T}^n$.

The *Ratner's orbit closure theorem* is a far generalization of this simple example. Just to give an initial ideia, let's look at the building blocks of this example.

- Observe that $\mathbb{R}^n$ is a **Lie group.**
- the subgroup $\mathbb{Z}^n$ is **discrete** and $\mathbb{T}^n$ is a manifold.

- the quotient space $\mathbb{R}^n/\mathbb{Z}^n$ is **compact.**
- the map $t \longmapsto tv$, which appears in the flow, is a **one-parameter subgroup** of $\mathbb{R}^n$.

Ratner's Theorem allows the following:

- $\mathbb{R}^n$ can be replaced by any **Lie group** G;
- the subgroup $\mathbb{Z}^n$ can be replaced by any **discrete** subgroup Γ such that the quotient G/Γ has **finite volume** and
- the map $t \longmapsto tv$ can be replaced by any **unipotent one-parameter subgroup** of G.

Unipotent Flows

Let G be a Lie group and Γ a subset of G. We define $G/\Gamma = \{x\Gamma : x \in G\}$ as the set of *left cosets* of Γ in G. We recall that a square matrix A is called *unipotent* when 1 is the only eigenvalue of A.

Definition 6. Let $\{u_t\}_{t\in\mathbb{R}}$ be a *one-parameter subgroup* of G, which acts on G/Γ by left multiplication. We say that $\{u_t\}_{t\in\mathbb{R}}$ is *unipotent* if the adjoint representation of every element in this subgroup is a unipotent matrix.

Example 1. Let $G = SL(2,\mathbb{R})$ be the group of 2x2 real matrices with determinant 1. Define $u, a : \mathbb{R} \to SL(2,\mathbb{R})$ by

$$u_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \text{ and } a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Easy computations show that $u_{t+s} = u_t u_s$ and $a_{t+s} = a_t a_s$ and so, $\{u_t\}_{t\in\mathbb{R}}$ and $\{a\}_{t\in\mathbb{R}}$ are one-parameter subgroups of $SL(2,\mathbb{R})$. But observe that $\{u_t\}_{t\in\mathbb{R}}$ is unipotent while $\{a_t\}_{t\in\mathbb{R}}$ is not unipotent (unless $t = 0$). For every subgroup $\Gamma \subset G$ consider the flows on G/Γ given by

$$\eta_t(x\Gamma) = u_t x\Gamma \text{ and } \gamma_t(x\Gamma) = a_t x\Gamma.$$

η_t is called *horocycle flow* on G/Γ and γ_t is called *geodesic flow* on G/Γ.

Definition 7. Let Γ be a subgroup of a Lie group G and let $\mathcal{B}$ be the *Borel-sigma algebra* on G, that is, the smallest sigma-algebra that contains the open sets of G. Every set in $\mathcal{B}$ is called a *measurable set.*

- A measure μ on G is *left-invariant* if $\mu(gA) = \mu(A)$ for all $g \in G$ and all measurable set $A \subset G$. Similarly, μ is *right-invariant* if $\mu(Ag) = \mu(A)$ for every $g \in G$ and all measurable set $A \subset G$.
- A subgroup Γ of G is said to be a *lattice* if Γ is discrete and G/Γ has finite Haar measure. □

Any Lie group G admits a (left) *Haar Measure*, that is, a left invariant Borel measure μ on G. Furthermore, this measure μ is unique up to a scalar multiple.

Theorem 8 (Ratner's Orbit Closure Theorem). *Let G be a connected Lie group and let Γ be a lattice in G. Let H be a connected Lie subgroup of G generated by one-parameter unipotent elements. Then for any $x \in G/\Gamma$ there exists a closed connected subgroup $P \supset H$ such that $\overline{Hx} = Px$ and Px admits a P-invariant probability measure.*

Observe that, when $H = \{u_t\}_{t\in\mathbb{R}}$, where $\{u_t\}_t$ is a one-parameter unipotent subgroup of G, the unipotent flow is given by the action of H by left on G/Γ, that is, for every $x = g\Gamma$ we have a flow $\phi_t(x) = \phi_t(g\Gamma) = u_t g\Gamma$. From this we can see $\overline{Hx} = \overline{\{\phi_t(g\Gamma) : t \in \mathbb{R}\}}$. The theorem says that the closure of this orbit is, in some sense, a "nice" geometric subset of G/Γ.

From example 1, we can see that the horocycle flow is unipotent and we can apply the Orbit Closure theorem. In fact, one can show that the closure of every η_t-orbit is dense in the entire space G/Γ.

On the other hand, the geodesic flow is not unipotent and Ratner's theorem does not apply. Indeed, it can be shown that some γ_t orbits are very far from being nice geometric subsets of G/Γ. For example, some orbits are fractals. More specifically, for some orbits, if O is the closure of the orbit, then some neighborhood (in O) of a point in O is homeomorphic to $C \times \mathbb{R}$ where C is the Cantor set.

2.2 Measure-theoretic Versions of Ratner's Theorem

Now, we try to connect the orbit closure theorem with some measure theoretical versions. For unipotent flows, Ratner's Orbit Closure Theorem says that the closure of each orbit is a nice geometrical subspace of G/Γ. In fact, as we will see, the orbit turns out to be *uniformly distributed* in its closure.

Let ϕ_t be the flow $\phi_t([x]) = [x + tv]$ on $\mathbb{T}^n$ defined by some vector $v \in \mathbb{R}^n$. Let μ be the normalized Lebesgue measure on $\mathbb{T}^n$, that is, with $\mu(\mathbb{T}^n) = 1$.

- Assume $n = 2$ and take $v = (a, b)$. It is a well known fact that if a/b is irrational every orbit $\phi_t(x)$ is dense in $\mathbb{T}^2$. In fact, every orbit is uniformly distributed in $\mathbb{T}^2$: if B is some nice open subset of $\mathbb{T}^2$, then the amount of time that each orbit spends in B is proportional to the area of B. More precisely, for each $x \in \mathbb{T}^2$, and letting λ be the Lebesgue measure on $\mathbb{R}$, we have

$$\frac{\lambda(\{t \in [0,T] : \phi_t(x) \in B\})}{T} \longrightarrow \mu(B) \quad \text{as} \quad T \longrightarrow \infty.$$

More general, for any continuous function f on $\mathbb{T}^2$, we obtain

$$\lim_{T\to\infty} \frac{\int_0^T f(\phi_t(x))dt}{T} = \int_{\mathbb{T}^2} f d\mu.$$

- Assume now $n = 3$ and take $v = (a, b, 0)$ with a/b irrational. In this case, the orbits are not dense and not uniformly distributed on $\mathbb{T}^3$ (with respect to the usual Lebesgue measure on $\mathbb{T}^3$). Instead, one can show that each orbit is uniformly distributed on some subtorus of $\mathbb{T}^3$: given $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, let μ_2 be the Haar measure on the horizontal 2-torus $\mathbb{T}^2 \times \{x_3\}$ that contains x. Then,

$$\lim_{T\to\infty} \frac{\int_0^T f(\phi_t(x))dt}{T} = \int_{\mathbb{T}^2\times\{x_3\}} f d\mu_2.$$

  In general, for any n, $v \in \mathbb{R}^n$ and any $x \in \mathbb{T}^n$, there is a subtorus S with Haar measure μ_S such that

$$\lim_{T\to\infty} \frac{\int_0^T f(\phi_t(x))dt}{T} = \int_S f d\mu_S.$$

This example generalizes to all unipotent flows:

Theorem 9 (**Ratner's Equidistribution Theorem**). *Let G be a connected Lie group, Γ any lattice in G and ϕ_t any unipotent flow on G/Γ. Then, each ϕ_t-orbit is uniformly distributed on its closure, that is, there exist a closed connected subgroup S of G such that*

$$\lim_{T\to\infty} \frac{\int_0^T f(\phi_t(x))dt}{T} = \int_{Sx} f d\mu_S,$$

for every continuous function on G/Γ, where μ_S is a S-invariant probability measure on Sx.

This theorem also yields some kind of classification of the ϕ_t-invariant probability measures.

Definition 10. Let X be a metric space, ϕ_t a continuous flow on X and μ a measure on X. We say that μ is ϕ_t-**invariant** if $\mu(\phi_t(A)) = \mu(A)$ for every Borel subset A of X and every $t \in \mathbb{R}$. We also say that μ is **ergodic** if μ is ϕ_t-invariant and for every ϕ_t-invariant Borel subset $A \subset X$ we have $\mu(A) = 0$ or $\mu(X - A) = 0$.

The following classical result from Measure Theory implies that we can study any invariant measure by considering only the ergodic invariant ones.

Theorem 11 (**Ergodic Decomposition Theorem**). *Let G be a Lie Group acting smoothly on $X = \Gamma/G$, where Γ is a lattice on G. Let μ be a G-invariant measure on X.Then there is a measure space (Y, ν) and a partition of X into G-invariant subsets X_y, $y \in Y$, and measures μ_y on X_y such that:*

1. *For any measurable subset $A \subset X$, we have that $A \cap X_y$ is measurable w.r.t. μ_y for ν-almost every $y \in Y$ and $\mu(A) = \int_Y \mu_y(A \cap X_y) d\nu(y)$.*
2. *For ν-almost every $y \in Y$, the action of G on X is ergodic w.r.t. the measure μ_y.*

Proof. See [8] or [10].

□

Theorem 12 (**Ratner's Measure Classification Theorem**). *Let G be a connected Lie group, Γ a lattice in G, ϕ_t a unipotent flow on G/Γ. Then, every ergodic ϕ_t-invariant probability measure is of the form μ_S, for some x and some subgroup S as in Theorem 9.*

The historical development occurred in the opposite direction: the closure orbit theorem and the equidistribution theorem were obtained from the measure classification theorem. This reveals something interesting: the knowledge of invariant measures can lead to information about closures of orbits.

The next proposition illustrates the connection between invariant measures and closures of orbits in this context.

Definition 13. Let ϕ_t be a continuous flow on a metric space X with σ-algebra Σ.

- We say that ϕ_t is **minimal** if every orbit is dense in X.
- ϕ_t is **uniquely ergodic** if there is a unique ϕ_t-invariant probability measure on X.
- The **support** of a measure μ on X is defined as the set

$$\operatorname{supp}(\mu) := \overline{\{A \in \Sigma : \mu(A) > 0\}}.$$

Proposition 14. *Let G be a connected Lie group, Γ a lattice in G, such that G/Γ is compact, and ϕ_t any unipotent flow on G/Γ. If ϕ_t is uniquely ergodic, then ϕ_t is minimal.*

Proof. Suppose that some orbit $\phi_{\mathbb{R}}(x)$ is not dense in G/Γ. We will show that the G-invariant measure μ_G is not the only ϕ_t-invariant probability measure on G/Γ.

Let Ω be the closure of $\phi_{\mathbb{R}}(x)$. Then, Ω is a compact ϕ_t-invariant subset of G/Γ. One also can see that there is a ϕ_t-invariant probability measure μ on G/Γ that is supported on Ω. Since

$$\text{supp}(\mu) \subset \Omega \subset G/\Gamma = \text{supp}(\mu_G),$$

we have that $\mu \neq \mu_G$. Hence, there are at least two different ϕ_t-invariant probability measures on G/Γ, and so ϕ_t is not uniquely ergodic.

□

We can obtain the Orbit Closure theorem from the Measure Classification theorem as follows:

By the Measure Classification Theorem, each $\phi_t(x)$ is uniformly distributed on its closure with respect to some invariant probability measure. In particular, there exists a closed connected subgroup P of G such that

$$\frac{\lambda(\{t \in [0,T] : \phi_t(x) \in B\})}{T} \longrightarrow \mu_P(B) \quad \text{as} \quad T \longrightarrow \infty,$$

for every open subset B of Px, where μ_P is a P-invariant probability measure on Px.

Suppose there exists a nonempty open subset B of Px such that $\phi_t(x) \cap B = \emptyset$ for every $t \in \mathbb{R}$. So, $\lambda(\{t \in [0,T] : \phi_t(x) \in B\}) = 0$, for every $T > 0$. From this we obtain that $\mu_P(B) = 0$, a contradiction. Hence, the orbit $\phi_t(x)$ visits every open subset of Px and $\overline{Ux} = \overline{\{\phi_t(x) : t \in \mathbb{R}\}} = Px$.

respective

An Outline of the proof of Ratner's Measure Classification Theorem for $G = SL(2, \mathbb{R})$

Now, we give the main ideas used in the proof of a version that appeared in [14], when Ratner proved the Raghunathan's Conjectures for $G = SL(2, \mathbb{R})$. The proof is a bit technical and we do not intend to prove every detail. The ideas in this approach involve the core strategy used by Ratner in the general case.

Theorem 15. *Let Γ be a discrete subgroup of $G = SL(2, \mathbb{R})$, U a unipotent subgroup of G and μ an ergodic U-invariant Borel probability measure on Γ/G. Then either 1) μ is supported on a periodic orbit of U or 2) Γ is a lattice and μ is G-invariant.*

Proof. Let the following subgroups be given:

$$U = \{U_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R}\},$$

$$A = \{A_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} : t \in \mathbb{R}\},$$

$$H = \{H_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{R}\}.$$

Since matrices can be written in upper-triangular form, every unipotent element of $SL(2, \mathbb{R})$ is conjugate to U_t for some $t \in \mathbb{R}$. These subgroups satisfy the following important relations:

$$U_s A_t = A_t U_{se^{-2t}},$$

$$H_s A_t = A_t U_{se^{2t}},$$

for every $s, t \in \mathbb{R}$.

Now we form the subgroups $W = AH$ and $B = AU$ with their neighborhoods

$$W(\delta) = \{A_\tau H_b : |\tau| < \delta, |b| < \delta\},$$

$$B(\delta) = \{A_\tau U_s : |\tau| < \delta, |s| < \delta\}.$$

Let $y = xW(\delta)$, that is, $y = xA_\tau H_b$ for some $|\tau| < \delta, |b| < \delta$. If $\delta > 0$ is sufficiently small, for any $y = xA_\tau H_b \in xW(\delta)$ and any $0 \leq s \leq 1$, there is a unique function $\alpha(y, s)$ that is strictly increasing in s, continuous in (y, s) and satisfying the condition $\alpha(y, 0) = 0$, such that $yU_{\alpha(y,s)} \in xU_s W(10\delta)$. Solving the system of linear equations obtained, we can see that

$$\alpha(y, s) = \frac{s}{e^{2\tau} - sb}$$

and that $yU_{\alpha(y,s)} = xU_s A_{\tau(y,s)} H_{b(y,s)}$, where

$$\tau(y, s) = \ln(e^\tau - bse^{2\tau})$$

$$b(y, s) = b(1 - bse^{-2\tau})$$

are the respective coordinates in the directions of A and H. As can be seen, these functions are defined only for $bs < e^{2\tau}$, since $\alpha(y, s)$ will have no solution outside this interval. This tells us that the divergence between points close to each other in some W-leaf will be very slow in the W-direction, but proportional to time in the U-direction. In particular, it says that if $b = 0$, then the two orbits will diverge only in the U-direction.

If however $b \neq 0$, then there will be a point in time $s_\theta > 0$, such that there is a $y \in xW(\delta)$ where $yU_\mathbb{R}$ does not intersect $xU_\mathbb{R}W$. The crucial part here is that the critical point in time may be made arbitrarily large by making δ small.

In the following, we state the so-called R-property, an important technical argument which appears originally in [11]. It captures the behavior of this critical point in time more exactly and plays a crucial role in the proof of Ratner's measure-classification theorem.

Lemma 16. ***(R-property)*** *There exist constants* $0 < \eta < 1$ *and* $C > 1$ *such that if for some* $t > 1$

$$|\tau(y,t)| = \theta \text{ and } |\tau(y,s)| \leq \theta \text{ for } 0 \leq s \leq t,$$

where $y \in xW(\delta)$*,* $0 < \delta < \theta/10$*, then*

$$\theta/2 \leq |\tau(y,s)| \leq \theta, |b(y,s)| \leq C\theta/s$$

for all $s \in [(1-\eta)t, t]$*.*

Proof. See [12]. □

Now, let μ be an ergodic U-invariant probability measure on $X = \Gamma/G$ and let $\Lambda = \Lambda(\mu) = \{g \in G : \text{ the action of } g \text{ on } X \text{ preserves } \mu\}$. It is clear from the definition that $U \subset \Lambda(\mu)$. Also, one can show that $\Lambda(\mu)$ is a closed subgroup of G.

The rest of the proof will basically be split into two cases: the one where μ is also A-invariant and the one where μ is not A-invariant.

Part 1: μ is not A-invariant

In this part, we give only some sketch of the proof. A complete and detailed proof can be found in [14].

Lemma 17. *If* $A \not\subset \Lambda$*, then there is an* $x \in X$ *such that* μ *is supported on the closed (periodic) orbit* xU*.*

Proof. Since the action of U is ergodic, there is a set $F \subseteq X$, $\mu(F) > 0$, such that, after some sufficiently long time, the U-orbit of every $x \in F$ will have spent almost of its time in some compact set K with almost full measure (say $\mu(K) > 0.99$). The idea is to use the R-property (Lemma 16) to obtain a small neighborhood $N(x) \cap F$ of some point $x \in F$ that looks like some small piece of the orbit xU, say $xU(\xi)$. This gives us that $xU(\xi)$ has positive measure and we can take a sufficiently large finite union

$$P := \{xU_s : -\xi \leq s \leq N\},$$

ensuring that $\mu(P) = 1$. Since, $PU_{\mathbb{R}}$ does not change the measure, the orbit must be periodic.

□

Part 2: μ is A-invariant

Here we give a more detailed proof, also following [14].

Lemma 18. *Suppose that $A \subset \Lambda$. Then, $\Gamma = SL(2, \mathbb{Z})$ is a lattice and μ is G-invariant.*

Proof. Let f be a continuous function on X with compact support. Since the action of A on (X, μ) is ergodic, there exists a set $C_f \subseteq X$, consisting of points $y \in X$ such that

$$S_{f,n}(y) = \frac{1}{n} \sum_{i=0}^{n-1} f(yA_{-i}) \longrightarrow f_\mu = \int_X f d\mu,$$

that is, C_f is a set of full μ-measure.

Since H-orbits are the contracting horocycles for geodesics in the negative direction in time, we see that for any $z \in yH$, $d_X(yA_{-n}, zA_{-n}) \longrightarrow 0$ when $n \longrightarrow \infty$, where d_X is some metric in X. Since f is uniformly continuous and $S_{f,n}(y) \longrightarrow f_\mu$, it follows that $S_{f,n}(z) \longrightarrow f_\mu$, and from this $C_f H = C_f$.

Now we need to prove that C_f is of full ν-measure. Let's start first considering the neighborhood

$$O_\delta(x) = xB(\delta/2)H(\delta) \cap xH(\delta/2)B(\delta)$$

for some sufficiently small $\delta > 0$, and the decomposition of μ on this neighborhood into conditional measures $\mu_y(E) = \mu(yB(\delta/2) \cap E)$ on the leaves $yB(\delta/2)$, $y \in xH(\delta)$. Since μ is B-invariant, almost every μ_y is also B-invariant. Hence, almost every μ_y is the Lebesgue measure on $yB(\delta/2)$.

Since C_f is of full μ-measure and

$$C_f \cap xB(\delta/2) = C_f h \cap xB(\delta/2)h = C_f \cap xB(\delta/2)h$$

for every $h \in H$, it follows that $C_f \cap O_\delta(x)$ has the same Lebesgue measure as $O_\delta(x)$. Since ν is the Lebesgue measure up to a constant, C_f must be of full ν-measure.

Now, let's assume that f is nonnegative and $f_\mu > 0$. By Fatou's lemma, we obtain

$$f_\mu \nu(X) = f_\mu \nu(C_f) = \int_{C_f} f_\mu d\nu \leq \lim_{n \to \infty} \int_{C_f} S_{f,n} d\nu = \int_{C_f} f d\nu = \int_X f d\nu < \infty.$$

To conclude, we wish to prove that $\mu = \nu/\nu(X)$. By Lebesgue's dominated convergence theorem, we obtain that

$$f_\nu = \int_X f d\nu = \int_{C_f} f d\nu = \int_{C_f} S_{f,n} d\nu \longrightarrow \int_{C_f} f_\mu d\nu = f_\mu \nu(X)$$

for every uniformly continuous function with compact support, which implies $\mu = \nu/\nu(X)$ as desired.

□

3 The Oppenheim Conjecture

3.1 Quadratic forms

Let

$$Q(x_1, ..., x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

be a *quadratic form* in n variables with $a_{ij} \in \mathbb{R}$. We always assume that Q is **indefinite**, that is, after a change of variables, Q can be expressed as

$$Q_p^{\#}(y_1, ..., y_n) = \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{n} y_i^2$$

for some $1 \le p < n$. Alternatively, we can se indefinite forms as forms that takes both positive and negative values.

Examples: $Q(x, y) = x^2 - 3xy + y^2$ **is** indefinite but $Q'(x, y) = x^2 - 2xy + y^2$ is **not** indefinite.

Proposition 19. *If a quadratic form Q is a multiple of a form with rational coefficients, then the set of values $Q(\mathbb{Z}^n)$ is a discrete subset of $\mathbb{R}$.*

Proof. Let $S(x_1, ..., x_n) = \sum_{i,j} b_{ij} x_i x_j$ be a rational form, that is, a form with rational coefficients $b_{ij} = \frac{p_{ij}}{q_{ij}}$. Taking $\alpha = \prod_{i,j} q_{ij}$ we can write S as

$$S(x_1, ..., x_n) = \frac{1}{\alpha} \left(\frac{p_{11}\alpha}{q_{11}} \right) x_1^2 + ... + \frac{1}{\alpha} \left(\frac{p_{nn}\alpha}{q_{nn}} \right) x_n^2.$$

Since q_{ij} divides α for every i, j we have that $\frac{p_{ij}\alpha}{q_{ij}} \in \mathbb{Z}$ for every i, j. Then, $\alpha S(\mathbb{Z}^n) \subset \mathbb{Z}$ which implies that $S(\mathbb{Z}^n)$ is a discrete set. If a quadratic form Q is a multiple of a rational form S, $Q(x_1, ..., x_n) = \beta S(x_1, ..., x_n)$ for some $\beta \in \mathbb{R}$ and it follows that $Q(\mathbb{Z}^n)$ is also a discrete subset of $\mathbb{R}$.

□

Oppenheim Conjecture (1929). *Suppose Q is a indefinite quadratic form that is not proportional to a rational form and $n \geq 5$. Then $Q(\mathbb{Z}^n)$ is dense in the real line.*

The conjecture was extended by Davenport in 1946:

Oppenheim-Davenport Conjecture (1946). *Suppose Q is a indefinite quadratic form that is not proportional to a rational form and $n \geq 3$. Then $Q(\mathbb{Z}^n)$ is dense in the real line.*

This result was proved by Margulis in 1986 and today is known as Margulis Theorem:

Theorem 20. *(**Margulis Theorem - 1986**) If $n \geq 3$ and Q is an indefinite quadratic form not proportional to a rational form, then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

Before Margulis, Oppenheim conjecture was attacked by analytic number theory methods. In particular, it was proved for $n \geq 21$ and for diagonal forms for $n \geq 5$.

Margulis proved this in generality using techniques of ergodic theory with some dynamics on homogeneous spaces. In this work, we intend to prove this theorem using the *Ratner's orbit closure theorem.*

Failure of the Oppenheim Conjecture in dimension 2

Let $\alpha > 0$ be a real number such that $\alpha^2 \notin \mathbb{Q}$, for example $\alpha = \frac{1+\sqrt{5}}{2}$. Consider the quadratic form

$$Q(x_1, x_1) = x_1^2 - \alpha^2 x_2^2.$$

Proposition 21. *There exists $\epsilon > 0$ such that for all $x_1, x_2 \in \mathbb{Z}$, $|Q(x_1, x_2)| > \epsilon$.*

Proof. Suppose, by contradiction, that is not. Then for any $0 < \epsilon < 1$ there exist $x_1, x_2 \in \mathbb{Z}$ such that

$$|Q(x_1, x_2)| = |x_1 - \alpha x_2||x_1 + \alpha x_2| \leq \epsilon. \tag{1}$$

We can assume $x_2 \neq 0$. If $\alpha^2 > \epsilon$ then either $|x_1 - \alpha x_2| < \alpha$ or $|x_1 + \alpha x_2| < \alpha$, otherwise

$$\epsilon \geq |Q(x_1, x_2)| \geq \alpha^2 > \epsilon,$$

a contradiction. Without loss of generality, we take $|x_1 - \alpha x_2| < \alpha$, so $|x_1 - \alpha x_2| < \alpha |x_2|$. From this,

$$|x_1 + \alpha x_2| = |2\alpha x_2 + (x_1 - \alpha x_2)| \geq 2\alpha |x_2| - |(x_1 - \alpha x_2)| \geq \alpha |x_2|. \tag{2}$$

Substituting (2) into (1) we obtain

$$\left|\frac{x_1}{x_2} - \alpha\right| \leq \frac{\epsilon}{\alpha}\frac{1}{|x_2|^2} \tag{3}$$

But since α is quadratic irrational ($\alpha \notin \mathbb{Q}$), there exist $c_0 > 0$ such that for all $p, q \in \mathbb{Z}$, $|\frac{p}{q} - \alpha| \geq \frac{c_0}{q^2}$. This is a contradiction to (3) with $\epsilon < c_0\alpha$.

□

The main key of the proof is the connection with the Ratner's Theorem considering the special orthogonal group of the quadratic form Q.

Definition 22. Let Q be a quadratic form in n variables.

- $SO(Q) = \{h \in SL(n, \mathbb{R}) : Q(vh) = Q(v), \forall v \in \mathbb{R}^n\}$ is called the *special orthogonal group* of Q.
- As a special case, we write $SO(m, n)$ as the special orthogonal group of $SO(Q_{m,n})$, where

$$Q_{m,n}(x_1, ..., x_{m+n}) = x_1^2 + ... + x_m^2 - x_{m+1}^2 - ... - x_{m+n}^2.$$

- Furthermore, we write $SO(m)$ to denote $SO(m, 0)$ which is equal to $SO(0, m)$.
- We set $SO(Q)^o$ to be the connected component of $SO(Q)$ that contains the identity element e. □

Outline of the Proof of the Margulis' Theorem

We assume that Q has only three variables. Since Q is indefinite, Q can be written as $Q_{2,1}$ or $Q_{1,2}$. Observe that $Q_{1,2}$ and $Q_{2,1}$ differ by an overall sign and we take Q_0 as the standard quadratic form $Q_{2,1}(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$. Then our arbitrary indefinite quadratic form Q is conjugate to $\pm Q_0$, that is, there exist $g \in SL(3, \mathbb{R})$ and $\lambda \in \mathbb{R}, \lambda > 0$, such that $Q = \lambda Q_o \circ g$. We can note that $SO(Q)^o = g.H.g^{-1}$, where $H = SO(Q_0)^o$ which is $SO(2, 1)^o$. As we will see, $H \approx SL(2, \mathbb{R})$ is generated by unipotent elements and since $SL(3, \mathbb{Z})$ is a lattice in $SL(3, \mathbb{R})$ we can apply Ratner's orbit closure theorem.

The Ratner's theorem says that:

- there exist a closed connected subgroup $P \supset H$;
- $\overline{Hg} = Pg$;
- there is an invariant probability measure on Pg;

One can show that there are only two possibilities for a closed, connected subgroup of $G = SL(3, \mathbb{R})$ containing $H = SO(Q_0)^o$: $P = H$ or $P = G$. We are considering the two cases separately.

Case 1: Assume $P = G = SL(3, \mathbb{R})$. In this case, $\Gamma g H$ is dense in G.

$$\begin{aligned} Q(\mathbb{Z}^3) &= Q_0(\mathbb{Z}^3 g) \quad \text{(definition of g)} \\ &= Q_0(\mathbb{Z}^3 \Gamma g) \quad (\mathbb{Z}^3 = \mathbb{Z}^3 \Gamma) \\ &= Q_0(\mathbb{Z}^3 \Gamma g H) \quad (\text{definition of } H = SO(Q_0)^o) \\ &\text{is dense in } Q_0(\mathbb{Z}^3 G) \quad (Q_0 \text{ is continuous}) \\ &= Q_0(\mathbb{R}^3/\{0\}) \quad (v.G = \mathbb{R}^3/\{0\} \text{ for all } v \neq 0) \\ &= \mathbb{R}. \end{aligned}$$

Case 2: $P = H$.

In this case, we want to prove that Q is proportional to a rational form. Here we need some standard results of algebraic groups.

Definition 23. A subset $H \subset SL(l, \mathbb{R})$ is called *Zariski closed* if there exist a subset $S \subset \mathbb{R}[x_{11}, ..., x_{ll}]$ such that $H = \{g \in SL(l, \mathbb{R}) : Q(g) = 0 \quad \forall Q \in S\}$, where we understand $Q(g)$ to denote the value obtained by substituting the matrix entries g_{ij} into the variables x_{ij}.

We call $\overline{\overline{H}}$ the *Zariski closure* of H, that is, the unique smallest Zariski closed set containing H.

We can see set as Zariski closed "if the matrix entries are characterized by polynomials.

Lemma 24. *(Borel density theorem) Let $H \subset SL(l, \mathbb{R})$ be a closed subgroup and let Γ be a lattice in H. Then, the Zariski closure $\overline{\overline{\Gamma}}$ of Γ contains every unipotent element of H.*

Lemma 25. *Let C be a subset of $SL(l, \mathbb{Q})$. Then $\overline{\overline{C}}$ is defined over $\mathbb{Q}$.*

Lemma 26. *For a nondegenerate quadratic form Q, $SO(Q)$ is defined over $\mathbb{Q}$ if and only if Q is proportional to a form with rational coefficients.*

The proofs of these lemmas can be found in [15].

From those lemmas we can proceed with the proof of case 2. If $P = H$, we have $gH = gP$ and by the Ratner's orbit closure theorem, gH has a finite invariant measure. Therefore, $\Gamma_g := \Gamma \cap (gHg^{-1}) = SL(3, \mathbb{Z}) \cap (gHg^{-1})$ is a lattice in $gHg^{-1} = SO(Q)^o$. Since H is generated by unipotent elements and H is closed because is equal to P, Borel density theorem implies that $SO(Q)^o$ is contained in the Zariski closure $\overline{\overline{\Gamma_g}}$ of Γ_g. Since $\Gamma_g \subset \Gamma = SL(3, \mathbb{Z}) \subset SL(3, \mathbb{Q})$, by Lemma 25 we have that $\overline{\overline{\Gamma_g}}$ is defined over $\mathbb{Q}$. But, since $SO(Q)^o \subset \overline{\overline{\Gamma_g}}$, by Lemma 26 we obtain that Q is proportional to a form with rational coefficients.

Then, the Margulis' theorem is proved for $n = 3$.

The argument can be generalized to any natural n as follows:

Suppose Q is a quadratic form on $\mathbb{R}^n$ satisfying the hypothesis of Margulis' Theorem. One can show that there exist $v_1, v_2, v_3 \in \mathbb{Z}^n$, such that the quadratic form $\overline{Q}$ on $\mathbb{R}^3$, defined as $\overline{Q}(x_1, x_2, x_3) = Q(x_1v_1+x_2v_2+x_3v_3)$, also satisfies the hypothesis of the theorem. Hence, $Q(\mathbb{Z}^n)$ is dense on the real line, as desired.

□

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